

AN APPLICATION OF MOORE'S CROSS-RATIO GROUP TO THE SOLUTION OF THE SEXTIC EQUATION*

BY

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Introduction.

By making use of the fact that all the double ratios of n things can be expressed rationally in terms of a properly chosen set of $n - 3$ double ratios, E. H. MOORE‡ has developed the "cross-ratio group," $C_{n!}$, a Cremona group in S_{n-3} of order $n!$ which is isomorphic with the permutation group of n things. H. E. SLAUGHT§ has discussed the $C_{5!}$ in considerable detail. Both $C_{5!}$ and $C_{6!}$ have been listed by S. KANTOR.||

I have already pointed out¶ that $C_{n!}$ defines a form-problem which I shall denote by $PC_{n!}$. The solution of $PC_{n!}$ carries with it the solution of the n -ic equation, and I have worked out in detail the application to the quintic equation. After the adjunction of the square root of the discriminant of the n -ic, a new form-problem, $PC_{n!/2}$, can be enunciated whose underlying group is the even subgroup $C_{n!/2}$ of $C_{n!}$. It appears in the particular case, $n = 5$, that the solution of $PC_{5!/2}$ can be expressed rationally in terms of the solution of KLEIN's A -problem.** The transition from the one problem to the other is accomplished by a very direct process which is suggested by simple geometric considerations. Explicit formulæ are always desirable, and these have been lacking hitherto in the solutions of the quintic equation in terms of the A -problem. By introducing $PC_{5!/2}$ as an intermediate stage I have obtained such formulæ from the long known invariant theory of the binary quintic.

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‡ American Journal of Mathematics, vol. 22 (1900), p. 279.

§ American Journal of Mathematics, vol. 22 (1900), p. 343; and vol. 23 (1901), p. 99.

|| See his *Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene*, Berlin (1895), p. 105, Type XIV; also *Theorie der Transformationen im R_3* , etc., *Acta Mathematica*, vol. 21 (1897), p. 1, in particular, p. 77, Type LI.

¶ These Transactions, vol. 9 (1908), p. 396, cited hereafter as c1.

** An explicit solution of the A -problem in terms of the icosahedral irrationality is found in c1, § 5.

It is my object in this paper to determine whether any material advantage is gained in the solution of the sextic equation by the introduction of the form-problem of the cross-ratio group of six things, $PC_{6!}$ or $PC_{6!/2}$. The main result in this direction is found in § 2, namely, that after the adjunction of the square root of the discriminant Δ of the given sextic S , the $PC_{6!/2}$ and therefore S itself has a rational resolvent sextic Σ . This resolvent Σ is of the general diagonal type, i. e., the fifth and third powers of the unknown are missing. This resolvent is not unknown,* and the main point of novelty here is the very natural way in which it is suggested by the $C_{6!}$.

In the first paragraph, the equations of generating substitutions of $C_{6!}$ are derived, particular sets of conjugate points are considered, the invariant spreads of the group are obtained, and the form-problems of $C_{6!}$ and $C_{6!/2}$ are set forth. The solution of the sextic S in terms of the solution of either of these form-problems is exhibited. In the second paragraph the solution of $PC_{6!/2}$ is expressed rationally in terms of the solution of the resolvent Σ . Some data concerning the history of Σ and its connection with a particular sextic Φ of MASCHKE are reviewed. These data are correlated to some extent by a geometric interpretation. A solution of $PC_{6!}$ by means of hyperelliptic modular functions also appears. In the third paragraph the reduction of the solution of Σ to the solution of the form-problem PG_{360} of the VALENTINER collineation group G_{360} in S_2 is outlined. The method there used applies equally well to the given sextic S ,† so that the advantage which arises from the introduction of Σ is due only to the absence of two of its coefficients. But the rational functions of the coefficients required for the reduction to PG_{360} are of necessity so complicated ‡ that this advantage is important.

§ 1. *The Invariants and Form Problems of $C_{6!}$.*

A sextic S with given roots z_1, z_2, \dots, z_6 can be transformed linearly in such a way that z_6 becomes ∞ and the other roots become y_1, y_2, \dots, y_5 respectively, where $\sum_1^5 y_i = 0$. The transformed roots y are determined to within a factor of proportionality. Their values in terms of the differences of the roots z are

$$\begin{aligned} \rho y_1 &= \overline{21} \overline{36} \overline{46} \overline{56} + \overline{31} \overline{26} \overline{46} \overline{56} + \overline{41} \overline{26} \overline{36} \overline{56} + \overline{51} \overline{26} \overline{36} \overline{46}, \quad ik = z_i - z_k, \\ \rho y_2 &= \overline{12} \overline{36} \overline{46} \overline{56} + \overline{32} \overline{16} \overline{46} \overline{56} + \overline{42} \overline{16} \overline{36} \overline{56} + \overline{52} \overline{16} \overline{36} \overline{46}, \\ (1) \quad \rho y_3 &= \overline{13} \overline{26} \overline{46} \overline{56} + \overline{23} \overline{16} \overline{46} \overline{56} + \overline{43} \overline{16} \overline{26} \overline{56} + \overline{53} \overline{16} \overline{26} \overline{46}, \\ \rho y_4 &= \overline{14} \overline{26} \overline{36} \overline{56} + \overline{24} \overline{16} \overline{36} \overline{56} + \overline{34} \overline{16} \overline{26} \overline{56} + \overline{54} \overline{16} \overline{26} \overline{36}, \\ \rho y_5 &= \overline{15} \overline{26} \overline{36} \overline{46} + \overline{25} \overline{16} \overline{36} \overline{46} + \overline{35} \overline{16} \overline{26} \overline{46} + \overline{45} \overline{16} \overline{26} \overline{36}. \end{aligned}$$

* See the references at the end of § 2.

† COBLE, *Mathematische Annalen*, vol. 70 (1911), p. 337; cited hereafter as c2.

‡ An estimate of the number and order of these functions is found in c2.

The differences of the transformed roots y are

$$(2) \quad \rho(y_i - y_k) = -5 \cdot i\bar{k} \bar{l}\bar{6} \bar{m}\bar{6} \bar{n}\bar{6} \quad (i, k, l, m, n=1, 2, \dots, 5).$$

The system of values y subject to the relation, $\sum y_i = 0$, can be interpreted as the homogeneous coördinates of a point y in S_3 . In this system of coördinates there is associated with the five reference planes, $y_i = 0$, a set of five points, each the polar point of one plane as to the tetrahedron of the other four. Denote these five points by p_1, p_2, \dots, p_5 ; their coördinates are $p_1(-4, 1, 1, 1, 1), \dots, p_5(1, 1, 1, 1, -4)$.

In the formulæ (1) let z_6 be a variable parameter, t . Then the point y describes a rational twisted cubic. As t takes the values z_1, z_2, \dots, z_6 , y takes the positions, p_1, p_2, \dots, p_5, y . Thus to construct the point y which corresponds to a given sextic S , we associate five of the roots of S to the points p and pass through these points a rational cubic such that their parameters p on the cubic are projective to the five roots of S . In this projectivity, the sixth root of S determines a sixth point y on the cubic curve. Since five roots of S can be associated with the five points in 720 ways, a sextic S determines a set of 720 points y . The change in y due to a permutation of the roots of S from some given order furnishes a transformation of Moore's group, $C_{6!}$. The 720 points y determined by S are a set of conjugate points under $C_{6!}$.

Let $T_{(\dots)\dots(\dots)}$ denote a permutation written in cycle form from the order z_1, z_2, \dots, z_6 to the order z'_1, z'_2, \dots, z'_6 . The same symbol will be used for the transformation which carries y into y' . Clearly the permutations of z_1, \dots, z_5 alone give rise to the permutation group of the five coördinates y , i. e., to the collineation group, C_{120} , which transforms the system of five planes y_i or the system of five points p_i into itself. The $C_{6!}$ can be generated by this group and an additional transformation, say $T_{(16)}$. To obtain this transformation we make use of formulæ (2).

$$(3) \quad \begin{aligned} \rho'(y'_1 - y'_2) &= -5 \cdot \bar{1}'\bar{2}' \bar{3}'\bar{6}' \bar{4}'\bar{6}' \bar{5}'\bar{6}' = -5 \cdot \bar{6}\bar{2} \bar{3}\bar{1} \bar{4}\bar{1} \bar{5}\bar{1} = \sigma \frac{1}{y_1 - y_2}, \\ \rho'(y'_1 - y'_3) &= -5 \cdot \bar{1}'\bar{3}' \bar{2}'\bar{6}' \bar{4}'\bar{6}' \bar{5}'\bar{6}' = -5 \cdot \bar{6}\bar{3} \bar{2}\bar{1} \bar{4}\bar{1} \bar{5}\bar{1} = \sigma \frac{1}{y_1 - y_3}, \\ \rho'(y'_1 - y'_4) &= -5 \cdot \bar{1}'\bar{4}' \bar{2}'\bar{6}' \bar{3}'\bar{6}' \bar{5}'\bar{6}' = -5 \cdot \bar{6}\bar{4} \bar{2}\bar{1} \bar{3}\bar{1} \bar{5}\bar{1} = \sigma \frac{1}{y_1 - y_4}, \\ \rho'(y'_1 - y'_5) &= -5 \cdot \bar{1}'\bar{5}' \bar{2}'\bar{6}' \bar{3}'\bar{6}' \bar{4}'\bar{6}' = -5 \cdot \bar{6}\bar{5} \bar{2}\bar{1} \bar{3}\bar{1} \bar{4}\bar{1} = \sigma \frac{1}{y_1 - y_5}, \end{aligned}$$

where

$$\sigma = 25\rho \cdot \bar{1}\bar{2} \bar{1}\bar{3} \bar{1}\bar{4} \bar{1}\bar{5} \bar{6}\bar{2} \bar{6}\bar{3} \bar{6}\bar{4} \bar{6}\bar{5}.$$

Thus $T_{(16)}$ is an involutory cubic transformation with the singular tetrahedron p_2, p_3, p_4, p_5 and the fixed point p_1 . Moreover any permutation which affects

z_6 gives rise to a cubic transformation with a singular tetrahedron whose vertices are found among the points p and with the same period as the permutation. The equations of the transformation can be obtained in precisely the same way as for $T_{(16)}$.

A point in general position is one of a set of 720 points which form a conjugate set under $C_{6!}$. If a point takes less than 720 positions under $C_{6!}$, it must be unaltered by a certain subgroup G of $C_{6!}$. To such a point there must correspond a class of projective binary sextics, each of which must be self-projective under a binary transformation group Γ which is isomorphic with G . If all the roots of the sextic are distinct, any permutation of them which can be effected by a binary projectivity defines the projectivity. If however equalities among the roots exist—to fix ideas let $z_1 = z_2$ —then to the identical projectivity there corresponds the identical permutation and also the permutation (z_1, z_2) . From the formulae (2) we find that the points y which correspond to sextics for which $z_i = z_k$ ($i, k = 1, 2, \dots, 5$) lie in the plane $y_i - y_k = 0$; those which correspond to sextics for which $z_i = z_6$ are directions about the point p_i . These five points and ten planes are a conjugate system of manifolds under $C_{6!}$, the locus of points for which the invariant Δ of S vanishes. Since we are interested primarily in the solution of S we shall assume that $\Delta \neq 0$. Then the groups G and Γ are simply isomorphic. The types of sextics S invariant under a group Γ have been tabulated by BOLZA.* From these types the systems of points in question are easily derived.

Because of the projective character of the transition from sextic S to point y it is clear that the locus of points y which map sextics S for which a certain invariant I of S vanishes is a surface, $M = 0$, invariant under $C_{6!}$. Conversely to every surface, $M = 0$, invariant under $C_{6!}$ there corresponds an invariant I of S . For if $M = 0$ is invariant under $C_{6!}$, M must be a symmetric function of y_1, y_2, \dots, y_5 . Since $\sum y_i = 0$, M is the leading coefficient of a covariant of the quintic, $(t_1 - y_1 t_2) \cdots (t_1 - y_5 t_2)$. Then $M = 0$ expresses that ∞ is a certain covariant point of y_1, \dots, y_5 , or that z_6 is a certain covariant point of z_1, \dots, z_5 . The condition that z_i be a similar covariant point of the other z 's is that $M = 0$ for all points conjugate to y , and this is satisfied, since $M = 0$ is invariant under $C_{6!}$. But this condition implies that the roots z satisfy a rational invariant relation, $I = 0$.

The invariant surfaces therefore are merely the invariants of the sextic

$$6t_2(t_1 - y_1 t_2) \cdots (t_1 - y_5 t_2), \quad \sum y_i = 0,$$

a sextic in a typical form employed by Clebsch (*Binäre Formen*, p. 351); the invariants of such forms are calculated there up to the sixth order. Another

* *Mathematische Annalen*, vol. 30 (1887), p. 546; *American Journal of Mathematics*, vol. 10 (1888), p. 47.

mode of procedure is to express the invariants of S in terms of the differences of the roots z . By identifying z_6 with ∞ and z_1, z_2, \dots, z_5 with y_1, y_2, \dots, y_5 we get the corresponding invariant surface from the coefficient of the highest power of z_6 . The index of this power is the degree n of I in the coefficients of S . The weight $3n$ of I is the order in the differences. But n differences in each term of I contain z_6 . Thus the order of the corresponding invariant surface M in y is $2n$. Since M must contain all the points to the same multiplicity k and also must be invariant under $T_{(16)}$, we find that $k = n$. Hence

All surfaces invariant under $C_{6!}$ are rational integral functions of a set of surfaces $M_4, M_8, M_{12}, M_{20}, M_{30}$ of orders 4, 8, 12, 20, 30 respectively and of multiplicities 2, 4, 6, 10, 15 respectively at the points p . These surfaces correspond to the members of a complete system of invariants of S of degrees 2, 4, 6, 10, 15 respectively.

The square of M_{30} can be expressed by means of the others; whence the general invariant surface M either is of order $4k$ with $2k$ -fold points at p_i or is the aggregate of such a surface and M_{30} . The discriminant Δ of degree ten corresponds to a surface of order twenty, which as we have seen must be the square of the ten planes on p . This surface has 12-fold points at p_i instead of the normal 10-fold points. This is accounted for the fact that the ∞^2 directions about p_i also map sextics for which $\Delta = 0$. In general if the multiplicity at p_i is greater than the normal multiplicity, $2k$, the ten planes are part of the invariant surface.

The simplest linear system of surfaces which is unaltered by $C_{6!}$ is the system of ∞^4 quadrics on the five points p . These quadrics map S_3 on a cubic surface in S_4 , whence there is among five linearly independent quadrics on p only a single cubic relation. There are

$$(n+1)(n+2)(n+3)(n+4)/4! - (n-2)(n-1)n(n+1)/4! = \frac{(n+1)(n^2+2n+2)}{2}$$

linearly independent forms of order n in these five quadratic forms. This is exactly the number.

$$\frac{1}{3!}(2n+1)(2n+2)(2n+3) - \frac{5}{3!}n(n+1)(n+2) = \frac{n+1}{2}(n^2+2n+2),$$

of linearly independent forms of order $2n$ with n -fold points at p_i . Hence every surface invariant under $C_{6!}$ can be expressed as an algebraic form with the five quadrics as variables. For purposes of manipulation however it is convenient to have a symmetrical set of quadrics on the points p_i . Since the five points are permuted by C_{120} , each member of such a set would be unaltered by a subgroup of C_{120} whose index is the number of members. This number should be five at least and we look for quadrics invariant under a metacyclic subgroup C_{20} of C_{120} whose index is 6. Selecting the particular subgroup generated by $T_{(12345)}$ and $T_{(2354)}$ we find that the quadric required is

$$(y_1 - y_2)(y_5 - y_3) + (y_2 - y_3)(y_1 - y_4) + (y_3 - y_4)(y_2 - y_5) \\ + (y_4 - y_5)(y_3 - y_1) + (y_5 - y_1)(y_4 - y_2).$$

By using the even permutations the following set of six conjugate quadrics is obtained :

$$\begin{aligned} 2A &= (25)(13) + (51)(42) + (14)(35) + (43)(21) + (32)(54), \quad (ik) = (y_i - y_k), \\ 2B &= (12)(53) + (23)(14) + (34)(25) + (45)(31) + (51)(42), \\ 2C &= (53)(41) + (34)(25) + (42)(13) + (21)(54) + (15)(32), \\ (4) \quad 2D &= (31)(45) + (24)(53) + (25)(41) + (15)(32) + (43)(21), \\ 2E &= (31)(24) + (12)(53) + (25)(41) + (54)(32) + (43)(15), \\ 2F &= (42)(35) + (23)(14) + (31)(52) + (15)(43) + (54)(21). \end{aligned}$$

There must exist among these quadrics a linear relation (due to the identities of the type $(12)(34) + (13)(42) + (14)(23) = 0$; this is necessarily symmetrical,

$$(5) \quad A + B + C + D + E + F \equiv 0.$$

The even permutation group of the y 's, a C_{60} , permutes the six quadrics as the icosahedral group permutes the diagonals of the icosahedron; the odd permutations permute the quadrics oddly into the quadrics with changed sign. The Cremona transformation $T_{(16)}$ permutes the quadrics in the order $(AE)(BC)(DF)$ to within a common outstanding factor. Thus to within common factors the quadrics are permuted in all possible ways by the C_{61} . Each quadric is unaltered by one of a second set of six conjugate subgroups of C_{61} of order 120. The cubic identity satisfied by these quadrics must be unaltered by C_{61} and therefore must be symmetric in A, \dots, F . Since $\Sigma A = 0$, this identity is

$$(6) \quad A^3 + B^3 + C^3 + D^3 + E^3 + F^3 \equiv 0.$$

Every invariant surface can be expressed in terms of these quadrics, and being invariant must be a symmetric or alternating rational integral function of them. Conversely every such function determines a surface invariant under C_{61} . For example, ΣA^5 does not vanish, since the quadrics satisfy only the relations, $\Sigma A = 0$ and $\Sigma A^3 = 0$. The odd permutations of C_{120} change the sign of ΣA^5 , hence it is an alternating invariant S of degree 5, i. e., to within a numerical factor, $\sqrt{\Delta}$. Also $\Pi(A - B)$ does not vanish and is unaltered by both the even and the odd permutations of C_{120} . It must be then the skew invariant of S of degree 15. The symmetric functions, ΣA^2 , ΣA^4 , and ΠA correspond to invariants of S of degrees 2, 4, and 6 respectively.

To identify the preceding results with known systems let us define, following

Clebsch and Gordan, some invariants and covariants of the sextic. Let

$$(7) \quad \begin{aligned} S &= a_x^6, & k &= k_x^4 = (ab)^4 a_x^2 b_x^2, \\ l &= l_x^2 = (ak)^4 a_x^2, & m &= m_x^2 = (kl)^2 k_x^2, & n &= n_x^2 = (km)^2 k_x^2, \\ A &= (ab)^6, & B &= (kk_1)^4, & C &= (kk_1)^2 (kk_2)^2 (k_1 k_2)^2, \\ D &= (ln)^2, & R &= (lm)(ln)(mn), & \sqrt{\Delta} &= \Pi(z_i - z_k) \quad (i < k). \end{aligned}$$

Adjusting the numerical coefficient in the formula for Δ given by Gordan (*Invarianten-Theorie*, p. 295) we have

$$(8) \quad -\frac{1}{2 \cdot 3^4} \Delta = 3 \cdot 2^7 A^5 - 3 \cdot 2^4 \cdot 5^3 A^3 B - 2^4 \cdot 5^4 A^2 C \\ + 3 \cdot 2 \cdot 5^5 (AB^2 + BC) + 3^2 \cdot 5^5 D.$$

Let the invariants A, B, C, D , and R be calculated for the sextic,

$$6t_2 \prod_1^5 (t_1 - y_i t_2),$$

and expressed symmetrically in terms of the quadrics A, \dots, F . These quadrics satisfy a sextic equation, $\Sigma = 0$, where

$$(9) \quad \Sigma = (Q - A)(Q - B) \dots (Q - F) = Q^6 + 15q_2 Q^4 + 15q_4 Q^2 + 6q_5 Q + q_6.$$

The coefficients of Σ are integral in $A, B, C, \sqrt{\Delta}$, and the square root of the discriminant of Σ is a numerical multiple of R . The result of the comparison is

$$(10) \quad \begin{aligned} 15q_2 &= \frac{5}{3} A, & 15q_4 &= \frac{5}{3} A^2 - \frac{5^3}{2 \cdot 3^2} B, \\ q_6 &= -\frac{5}{3^3} A^3 + \frac{5^3}{2 \cdot 3^3} AB + \frac{5^4}{3^4} C, & 6q_5 &= \frac{1}{3^3} \sqrt{\Delta}^4. \end{aligned}$$

Another set of fundamental invariants is defined by the equations,

$$(11) \quad \begin{aligned} \bar{A} &= \frac{5}{9} A, & \bar{B} &= \frac{1}{2 \cdot 9^2} (8A^2 - 75B), \\ \bar{C} &= \frac{5}{2 \cdot 9^3} (-2^3 \cdot 13A^3 + 3^2 \cdot 5^2 AB + 2 \cdot 3^2 \cdot 5^3 C), \end{aligned}$$

where \bar{B} and \bar{C} both vanish if S has a triple root. Then the sextic Σ has the particularly simple form

$$(12) \quad \Sigma = (Q^2 + \bar{A})^3 + 15\bar{B}Q^2 + \frac{1}{3^3} \sqrt{\Delta} Q + C = 0.$$

If we let

$$(13) \quad \sqrt{\Delta_2} = \Pi(A - B) \dots (A - F)(B - C) \dots (B - F) \dots (E - F),$$

then

$$(14) \quad \sqrt{\Delta_2} = -\frac{2^2 \cdot 5^{10}}{3^6} R.$$

The form problem of the $C_{6!}$ reads as follows:

(15) *Given the numerical values of A, B, C, Δ , invariant under $C_{6!}$, to find the ratios of the coördinates of a point y for which these quantities take the assigned values.*

The given values determine the numerical values of B/A^2 , C/A^3 , and Δ/A^5 , whence y is a meeting point of invariant surfaces of orders 8, 12 and 20. Apart from the points p of multiplicities 4, 6, and 10 respectively, these surfaces meet in $8 \cdot 12 \cdot 20 - 5 \cdot 4 \cdot 6 \cdot 10 = 720$ points which form a conjugate set under $C_{6!}$. Thus all the solutions of $PC_{6!}$ are obtained rationally from one solution by means of the transformations of $C_{6!}$.

If the numerical value which the product of the ten planes, $\sqrt{\Delta}$, takes for the point y be given it is not possible to determine a set of 360 points conjugate under $C_{6!/2}$ by means of three absolute invariants alone. As before, the known ratios B/A^2 , C/A^3 , and Δ/A^5 determine two sets of 360 conjugate points. If also the value R be given, the known ratios B/A^2 , C/A^3 , $R/(\sqrt{\Delta})^3$ determine three sets of 360 conjugate points, since the surfaces are of orders 8, 12, and 30 with 4-, 6-, and 15-fold points respectively at p_i . But there is an identity (Gordan, *Invarianten-theorie*, pp. 290-1) which expresses the square of $R/(\sqrt{\Delta})^3$ in terms of B/A^2 , C/A^3 and Δ/A^5 . Since this identity is cubic in Δ/A^5 , only one of the above three sets is contained in the two sets. Thus the form-problem of $PC_{6!/2}$ reads:

(16) *Given the numerical values of the invariants $A, B, C, \sqrt{\Delta}$, and R subject to the identity which connects them, to find the ratios of the coördinates y of one of the associated set of 360 points conjugate under $C_{6!/2}$.*

We have to show finally how the roots z of the given sextic S can be expressed rationally in terms of a solution, y , of $PC_{6!/2}$. First let us assume that $\Delta \neq 0$ and $R \neq 0$, for otherwise S can be solved by means of radicals. The invariants A, B, C, Δ , and R of S are calculated and a value of $\sqrt{\Delta}$ is adjoined. These values are identified with the numerical values in $PC_{6!/2}$. The solution of $PC_{6!/2}$ furnishes a sextic

$$S' = 6t_2(t_1 - y_1 t_2) \cdots (t_1 - y_5 t_2)$$

which is projective to S . If we denote the invariants and covariants of S' by primed letters, then

$$(17) \quad I_2'(lx)^2 = \rho I_2'^2(l't)^2, \quad I_2'(mx)^2 = \rho I_2'(m't)^2, \quad (nx)^2 = \rho(n't)^2.$$

Here I_2 is a non-vanishing invariant of weight 6 which can always be formed from A, B, C , and $\sqrt{\Delta}$ unless $A = B = C = 0$, in which case S is solvable by radicals. ρ is a power of the determinant of the projectivity that transforms S' into S . The determinant of the system (17) in the variables $x_1^2, 2x_1x_2$, and x_2^2 is $I_2^3 R$ and is not zero. From this system we can find the

values of x_1^2/ρ , $2x_1x_2/\rho$, and x_2^2/ρ in terms of the coefficients of S , of the coördinates y_1, \dots, y_5 and of t_1 and t_2 . Then from

$$z = \frac{\frac{1}{\rho}x_1^2}{\frac{1}{\rho}x_1x_2} \quad \text{or} \quad z = \frac{\frac{1}{\rho}x_1x_2}{\frac{1}{\rho}x_2^2}$$

we get the six roots z_1, z_2, \dots, z_6 by assigning the values $y_1, y_2, \dots, y_5, \infty$ to the ratio t_1/t_2 .

Hence the solution of S and that of PC_{61} or the solution of S after the adjunction of $\sqrt{\Delta}$ and that of $PC_{61/2}$ are rationally equivalent problems.

§ 2. The Sextic Resolvent Σ of $PC_{61/2}$.

The given quantities in $PC_{61/2}$ as formulated in (16) determine rationally the coefficients of the sextic Σ in (9). Also, according to (14), $\sqrt{\Delta_\Sigma}$ is rationally known from $PC_{61/2}$. The Galois group of Σ is of order 360 and is simply isomorphic with $C_{61/2}$. Since the roots of Σ are rational functions of the required coordinates y in $PC_{61/2}$, Σ is in a sense a rational resolvent of $PC_{61/2}$. In order to complete the connection between Σ and $PC_{61/2}$ we require formulæ which give the ratios of the coordinates y in terms of the roots A, \dots, F of Σ . From the formulæ (4) we get at once

$$\begin{aligned} \frac{1}{2}(A+B) &= (51)(42), & \frac{1}{2}(B+C) &= (34)(25), & \frac{1}{2}(C+E) &= (13)(42), \\ \frac{1}{2}(A+C) &= (53)(41), & \frac{1}{2}(B+D) &= (31)(45), & \frac{1}{2}(C+F) &= (54)(21), \\ (18) \quad \frac{1}{2}(A+D) &= (43)(21), & \frac{1}{2}(B+E) &= (12)(53), & \frac{1}{2}(D+E) &= (25)(41), \\ \frac{1}{2}(A+E) &= (32)(54), & \frac{1}{2}(B+F) &= (23)(14), & \frac{1}{2}(D+F) &= (24)(53), \\ \frac{1}{2}(A+F) &= (25)(13), & \frac{1}{2}(C+D) &= (15)(32), & \frac{1}{2}(E+F) &= (15)(43). \end{aligned}$$

From these pairs of differences the ratios of four linearly independent differences can be derived in a variety of ways. One verifies at once that

$$\begin{aligned} \rho(12) &= -(A+D)(B+E)(C+F), \\ \rho(13) &= (A+D)(B+E)(B+D), \\ (19) \quad \rho(14) &= (A+D)(A+C)(C+F), \\ \rho(15) &= (E+F)(B+E)(C+F), \end{aligned}$$

where

$$\rho = 8 \cdot (12)^2(34)(35)(45).$$

From these ratios of the differences we find the required ratios of the coördinates y ,

$$\begin{aligned}
 (20) \quad & 5y_1 = (12) + (13) + (14) + (15) \\
 & 5y_2 = -4(12) + (13) + (14) + (15) \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
 \end{aligned}$$

Hence the solution of Σ and the solution of $PC_{6\frac{1}{2}}$ are rationally equivalent problems.

Since the solutions of $PC_{6\frac{1}{2}}$ and the solution of S after the adjunction of $\sqrt{\Delta}$ are also rationally equivalent, it must be possible to exhibit Σ directly as a rational resolvent of S . This is accomplished by passing from the quadrics A, \dots, F to functions of the differences $(z_i - z_k)$ by means of formulae (2). Dropping the factor, $25\Pi_{i=1}^5(i6)$, from the six we obtain functions, A', \dots, F' , of weight 3 and of order 1 in a particular root. They arise from A, \dots, F by replacing each term $(ik)(lm)$ by the term $(ik)(lm)(n6)$. These new functions A', \dots, F' are the roots of a rational resolvent of S after adjunction of $\sqrt{\Delta}$. They have been used by previous writers on this subject, but in these earlier accounts the introduction of the functions seems to be somewhat fortuitous. Here their existence and properties have been shown to be direct consequences of quite elementary geometric group-theory. Moreover the inverse process of expressing the roots of S in terms of the roots of $\Sigma[(19), (20), (17)]$ is as readily suggested and accomplished by the use of familiar formulae.

The functions, A', \dots, F' of the six roots z seem to have been first used by Joubert,* who has noted their most striking properties and has calculated the equation which they satisfy. A similar study is found in a paper by H. W. Richmond.† Joubert's main interest in this sextic apparently lay in its utility as a resolvent of the quintic.‡

From a quite different point of view, H. Maschke§ developed another particular sextic,

$$(21) \quad \Phi = y^6 - 6F_8y^4 + 4F_{12}y^3 + 9F_8^2y^2 - 12F_{20}y + 4F_{24} = 0.$$

The coefficients F_i are quaternary forms of order i in the variables z_1, z_2, z_3, z_4 which are invariant (or which at most change in sign if $i \not\equiv 0 \pmod{8}$) under a group of linear substitutions of order 64.720. The group is obtained by effecting transformations of the periods on the Borchardt moduli, z_i , for the hyper-elliptic thetas of genus 2, where

$$\begin{aligned}
 (22) \quad & z_1 = \theta_5 = \vartheta_5(0, 0; 2\tau_{11}, 2\tau_{12}, 2\tau_{22}), & z_3 = \theta_{23} = \vartheta_{23}(0, 0; 2\tau_{11}, 2\tau_{12}, 2\tau_{22}), \\
 & z_2 = \theta_4 = \vartheta_4(0, 0; 2\tau_{11}, 2\tau_{12}, 2\tau_{22}), & z_4 = \theta_{01} = \vartheta_{01}(0, 0; 2\tau_{11}, 2\tau_{12}, 2\tau_{22}),
 \end{aligned}$$

* *Comptes Rendus* (1867), p. 1025 and p. 1080.

† *Quarterly Journal of Mathematics*, vol. 31 (1899), p. 57.

‡ See the historical account by KLEIN, *Ikosäeder*, II, 1, §§ 4-5, p. 148.

§ *Mathematische Annalen*, vol. 30 (1887), p. 496; in particular p. 506, (14).

The roots of $\Phi = 0$ are quartic forms in the z 's which I shall denote by Φ_A, \dots, Φ_F [see Maschke, l. c. (7) and (11)], each of which is unaltered by the same group, Γ_{64} , of linear substitutions or by an isomorphic group, Γ_{16} , of collineations. They satisfy the two identities

$$(23) \quad \sum \Phi_A \equiv 0, \quad \sum \Phi_A \Phi_B \Phi_C \Phi_D = \frac{1}{4} (\sum \Phi_A \Phi_B)^2.$$

On the other hand, Bolza* has obtained the values of the invariants of the binary sextic in terms of the zero values of the hyperelliptic thetas, $\vartheta_{ik}(0, 0; \tau_{11}, \tau_{12}, \tau_{22})$, associated with the sextic. The squares of the ten even thetas can be expressed as quadratic functions of the z 's and therefrom the invariants can be derived as functions of the forms F_i [Bolza, loc. cit., (30), (32); Maschke, loc. cit., (21) and (23)],

$$(24) \quad A^0 = \frac{\rho^2}{3 \cdot 5 \cdot 2^7} \frac{F_{12}^2 - F_{24}}{F_8 F_{12} - F_{20}}, \quad B^0 = \frac{\rho^4}{3^4 \cdot 5} F_8, \quad C^0 = -\frac{\rho^6}{3^6 \cdot 5} F_{12}, \dagger$$

$$\Delta^0 = \frac{2^8 \rho^{10}}{3^4} (F_8 F_{12} - F_{20}), \quad \rho = \frac{(2\pi i)^2}{\omega_{11} \omega_{22} - \omega_{21} \omega_{12}}.$$

Maschke ‡ has remarked further that by means of a Tschirnhausen transformation, $y = x^2 + \lambda x + \mu$, the general sextic can be transformed into $\Phi = 0$ by solving a quartic equation. Then the known values of F_i determine the absolute invariants formed from A^0, B^0, C^0, Δ^0 . If therefore the periods and the thetas be calculated for a normal hyperelliptic integral associated with a binary sextic which has these absolute invariants, values of $\sqrt{\rho} z_i$ in (24) are obtained and these in turn determine the roots Φ_A of $\Phi = 0$. Thus the general sextic is solved by means of hyperelliptic modular functions.

Commenting on this paper, F. Brioschi § notes that the sextic Φ can be written as

$$(25) \quad (y^3 - 3F_8 y + 2F_{12})^2 + 12(F_8 F_{12} - F_{20})y - 4(F_{12}^2 - F_{24}) = 0.$$

If we use Bolza's invariants and set $\xi = \rho^2 y/3$, this becomes

$$(26) \quad (\xi^3 - 5 \cdot 3^3 B^0 \xi - 10 \cdot 3^3 C^0)^2 + \frac{\Delta^0}{4^3} (\xi - 5 \cdot 2^7 A^0) = 0.$$

If

$$(27) \quad \begin{aligned} \xi &= 5 \cdot 2^7 A^0 - t^2, \\ T &= t^6 - 3 \cdot 5 \cdot 2^7 A^0 t^4 + 3 \cdot 5 (5 \cdot 4^7 A^{0^2} - 9 B^0) t^2 \\ &\quad + \frac{\sqrt{\Delta}}{8} t - 10 (5^2 \cdot 4^{10} A^{0^3} - 5 \cdot 12^3 A^0 B^0 - 3^3 C^0) = 0. \end{aligned}$$

* *Mathematische Annalen*, vol. 30 (1887), p. 478.

† The A^0, B^0, C^0, Δ^0 here refer to BOLZA's A, B^*, C^*, Δ .

‡ *Rendiconti della R. Accademia dei Lincei*, series 4 (1888), p. 181.

§ *Rendiconti della R. Accademia dei Lincei*, series 4 (1888), p. 183.

In later papers Brioschi* develops more fully the connection between these sextics and the theta moduli.

The sextic T is in Joubert's normal form. By comparing it with Σ in (12), the relations between the invariants of Bolza and those of §1 appear. Clearly the solution of T or Σ implies the solution of Maschke's sextic Φ and conversely. We shall find farther on [see (32)] the rational transformation inverse to (27).

The formulæ given above are illuminated by some geometric considerations allied with the papers of H. W. Richmond.† Denote the quadrics A by $A(y^2)$, their polarized forms by $A(yy')$; the quartics Φ_A by $\Phi_A(z^4)$, their polarized forms by $\Phi_A(z^3z')$. If we polarize the identity (6), $\Sigma A^3(y^2) \equiv 0$, we get the identity, $\Sigma A^2(y^2) A(yy') \equiv 0$. But this expresses that, for a given point y , the quadric, $\Sigma A^2(y^2) A(y^2) = 0$, has a double point at y . For given y' we have the condition that a quadric on the six points p_i and y' may have a double point at y . Hence

(28) *The equation, $\Sigma A^2(y^2) A(y'^2) = 0$, is, for given y , a quadric cone on the points p with vertex at y ; for given y' , the Weddle quartic surface with double points at the points p and y' .*

The quartics Φ_A are unaltered by a collineation Γ_{16} determined by a Kummer surface, and every such quartic is a linear combination of the Φ_A 's, say $\Sigma \lambda_A \Phi_A = 0$. Since $\Sigma \Phi_A = 0$ we can suppose that $\Sigma \lambda_A = 0$. The condition that $\Sigma \lambda_A \Phi_A = 0$ be a Kummer surface is then easily shown to be $\Sigma \lambda_A^3 = 0$. Hence $\Sigma A(y^2) \Phi_A = 0$ is for given point y a Kummer quartic. It is easily verified ‡ that the fundamental sextic of this Kummer quartic and the sextic $y'_1, \dots, y'_5, \infty$ are projective; whence

(29) *The Weddle quartic surface, $\Sigma A(y'^2) A^2(y^2) = 0$, and the Kummer quartic surface, $\Sigma A(y'^2) \Phi_A(z^4) = 0$, are birationally equivalent.*

If on the other hand we polarize the quartic identity (23) we get the new identity

$$\Sigma \Phi_A(z^3z') \Phi_B(z^4) \Phi_C(z^4) \Phi_D(z^4) - \frac{1}{2} [\Sigma \Phi_A(z^4) \Phi_B(z^4)] [\Sigma \Phi_A(z^3z') \Phi_B(z^4)] \equiv 0.$$

Interchanging z and z' we have the condition that the quartic surface in the variables z ,

$$(30) \quad \Sigma \Phi_A(z^4) \Phi_B(z'^4) \Phi_C(z'^4) \Phi_D(z'^4) - \frac{1}{2} [\Sigma \Phi_A(z'^4) \Phi_B(z'^4)] [\Sigma \Phi_A(z^4) \Phi_B(z'^4)] = 0,$$

may have a double point at z' . Being unaltered by Γ_{16} it must have 16 double points, whence

* *Acta Mathematica*, vol. 12 (1889), p. 83; *Annales de l'Ecole Normale Supérieure*, (3), vol. 12 (1895), p. 343.

† *Quarterly Journal of Mathematics*, vol. 31 (1899), p. 125; vol. 34 (1902), p. 117.

‡ The necessary formulæ are given by HUDSON, *Kummer's Quartic Surface*, pp. 81-2.

(31) *The quartic surface (30) is a Kummer quartic surface with double points at the 16 points conjugate to z' under Γ_{16} .*

The coefficients of $\Phi_A(z^4), \dots$, in (30) after being modified so that their sum is zero can be identified with the coefficients of $\Phi_A(z^4)$ in (29). Expressing each coefficient in terms of the symmetric functions of the Φ_A 's, i. e., the F_i , and of a particular Φ_A we find with little trouble that

$$(32) \quad A = -\Phi_A^3 + 3F_3\Phi_A - 2F_{12}.$$

This formula furnishes the Tschirnhausen transformation of the sextic Φ into the sextic Σ .

If we interpret the A 's as homogeneous point coordinates, and the Φ_A 's as space coordinates in S_4 , then $\Sigma A^3 = 0$ is the equation of a diagonal cubic surface, K_3 , whose reciprocal is a quartic envelope, R_4 . The equation of R_4 is precisely the quartic identity (23). Thus we have the particularly simple interpretation of (27) and (32):

(33) *The inverse Tschirnhausen transformations (27) and (32) from the Joubert to the Maschke normal form (and vice versa) represent the process of passing from point to tangent space (and vice versa) of the reciprocal loci K_3 and R_4 in the space S_4 .*

If the Maschke sextic be solved as already indicated, then the roots of Σ are obtained from (32). Consequently also the solution of PC_{61} in terms of hyperelliptic modular functions of genus two is known. Ultimately then the roots of the given sextic S are expressed in terms of such functions, i. e. in terms of transcendental functions of three variables, the moduli $\tau_{11}, \tau_{12}, \tau_{22}$. This however can hardly be accepted as a final solution. The PG_{360} contains only two independent variables and by the use of it in place of the modular functions an essential simplification is obtained.

§ 3. *The Reduction of PC_{612} to the VALENTINER Form-Problem.*

The even permutations which do not alter z_6 give rise to a collineation subgroup C_{60} of C_{612} . This subgroup is one of a set of six conjugate subgroups, $G_{60}^{(1)}, \dots, G_{60}^{(6)}$, the root z_i being associated with $G_{60}^{(i)}$. The quadrics A, \dots, F also are permuted in all possible even ways by C_{612} . Therefore C_{612} contains a second set of six conjugate subgroups, $G_{60}^{(A)}, \dots, G_{60}^{(F)}$, the quadric A being unaltered by $G_{60}^{(A)}$, etc. The same set of subgroups $G_{60}^{(A)}$ is defined by the functions A', \dots, F' of the roots z . If then we begin with the quadrics A, \dots, F and form from them six functions A'', \dots, F'' in the same way that the A 's are formed from z_i , these new functions will be unaltered respectively by a set of conjugate subgroups G_{60} which are different from $G_{60}^{(A)}, \dots, G_{60}^{(F)}$. The groups $G_{60}^{(A'')}, \dots, G_{60}^{(F'')}$ must be identical to within their order with the groups $G_{60}^{(1)}, \dots, G_{60}^{(6)}$, since C_{612} is known to contain only two

systems of conjugate icosahedral subgroups. Let us then name these functions $H^{(1)}, \dots, H^{(6)}$. Each H is of order 3 in the differences of A, \dots, F , of order 6 in the differences of the y 's and of maximum order 3 in a particular y . Also since H_6 is unaltered by $G_{60}^{(6)}$ or C_{60} it must be a seminvariant of the quintic with roots y of degree 3 and weight 6. The quintic has only one such covariant—its canonizant of order 3. Thus $H_6 = 0$ must be the condition that ∞ be a canonizant point of y_1, y_2, \dots, y_5 . Similarly $H_5 = 0$ is the condition that y_5 be a canonizant point of y_1, \dots, y_4, ∞ ; etc. The surfaces, $H_i = 0$, are sextic surfaces with triple points at p_i which satisfy relations similar to (5) and (6). Adjusting the numerical factor so that

$$\sum_{i=1}^6 H_i \equiv 0, \quad \sum_{i=1}^6 H_i^3 \equiv 0,$$

we find that

$$(36) \quad \begin{aligned} H_i = & y_i^6 - 3y_i^4 s_2^{(i)} + 2y_i^3 s_3^{(i)} - y_i^2 [(s_2^{(i)})^2 - 70s_4^{(i)}] + 8y_i s_2^{(i)} s_3^{(i)} \\ & + (s_2^{(i)})^3 + 8(s_3^{(i)})^2 - 20s_2^{(i)} s_4^{(i)} \quad (i=1, 2, 3, 4, 5); \end{aligned}$$

$$H_6 = -s_2^3 - 10s_3^2 + 20s_2 s_4,$$

where

$$s_k = \Sigma y_1 y_2 \cdots y_k, \quad s_k^{(i)} = s_k - y_i s_{k-1}^{(i)}, \quad s_0^{(i)} = 1.$$

The Valentiner group of 360 ternary collineations, G_{360} , is isomorphic with $C_{6/2}$ and has also two sets of conjugate icosahedral subgroups, $G_{60}^{(1)}, \dots, G_{60}^{(6)}$; and $G_{60}^{(A)}, \dots, G_{60}^{(F)}$. As invariants under the subgroups we have two sets of six conics, $(k_1 x)^2, \dots, (k_6 x)^2$; and $(ax)^2, \dots, (fx)^2$ [See C2, § 1]. If each conic be multiplied by its reciprocal, the two sets of six products are permuted by G_{360} without extraneous factors. Hence the forms

$$\sum_1^6 (k_1 x)^2 \cdot (k_1 k'_1 u)^2 \cdot H_1$$

and

$$(37) \quad \sum_a^f (ax)^2 \cdot (aa' u)^2 \cdot A$$

are *Kleinian forms*, i. e., algebraic forms in the three sets of variables x_1, x_2, x_3 ; u_1, u_2, u_3 ; y_1, y_2, \dots, y_5 , which are merely multiplied by a factor when the variables x and u are contragrediently transformed by the collineations of G_{360} and simultaneously the variables y are transformed by the corresponding substitutions of $C_{6/2}$. The correspondence between G_{360} and $C_{6/2}$ is established by associating H_i with $(k_i x)^2$. From these two forms we derive the covariant connex (1, 1)

$$(38) \quad \sum_1^6 \sum_a^f H_1 \cdot A \cdot (k_1 k'_1 a)^2 \cdot (k_1 aa') (k_1 x) (aa' u)$$

which is also a Kleinian form. This connex viewed as a collineation has three fixed points which can be separated by means of an "accessory" cubic equation. One of these fixed points is a so-called "covariant point" by means of which the solution of $PC_{6\frac{1}{2}}$ is reduced to PG_{360} .

Simpler algebraically than this direct transition from $PC_{6\frac{1}{2}}$ to PG_{360} is the solution of the resolvent Σ in terms of PG_{360} . This is accomplished exactly as outlined for the general sextic in $C2$ (in particular §4), the expressions being considerably shorter owing to the fact that in the case of Σ , $a_1 = a_3 = 0$. For example, the accessory cubic equation [$C2$, §4, (1), (2), (3), (4)] is

$$\sigma^3 + J\sigma - K = 0,$$

where

$$(39) \quad \begin{aligned} J &= \frac{1}{2} [15q_2^2 q_4 - \frac{1}{5} q_2 q_6 - 2q_4^2], \\ K &= \frac{1}{2} [-45q_2^2 q_4^2 + \frac{6}{5} q_2 q_4 q_6 + 8q_4^3 - \frac{1}{25} q_6^2]. \end{aligned}$$

From the preceding considerations I conclude that *as a matter of practical convenience* it seems advisable to effect the solution of the diagonal sextic resolvent Σ in terms of PG_{360} rather than the given sextic S . As the natural bond between S and Σ , the Cremona $C_{6\frac{1}{2}}$ must be introduced.

BALTIMORE, Oct. 1, 1910.